Large-Scale Epidemic Models and a Graph-Theoretic Method for Constructing Lyapunov Functions

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Large-scale epidemic models are used to describe disease transmission and spread in heterogeneous host populations, often structured by social or age groups, differential infectivity and susceptibility, and spatial heterogeneity and patchy environments. Large-scale models also arise when there are multiple strains of the pathogens, multiple types of hosts and diseases vectors. I will describe how to set up a multi-group model for HIV transmission and a multi-city model for Dengue fever.

Many, if not all, of the large-scale epidemic models can be investigated mathematically in the framework of coupled dynamical systems on directed graphs (a digraph), in which a simple dynamical system (a local model) is defined at each vertex of the graph, and a directed edge and its weight describes the influence from one vertex system to another (for instance, the influence can be cross-infection or movement of people), and the influences between a pair of vertices are not necessarily symmetric.

Analysis of large-scale epidemic models follows a similar procedure as that for simple models, except that it is technically more challenging. Derivation of the basic reproduction number R0 is typically handled by the method of van den Driessche and Watmough [1], which also ensures that the disease-free equilibrium P0 is locally asymptotically stable (disease dies out) when R0<1, and P0 is unstable (disease persists) when R0>1, and there exists an endemic equilibrium P*. Uniqueness of the endemic equilibrium and its local stability are difficult to analyze due to the large number of equations. The global attractivity of the equilibria becomes highly nontrivial. Constructing Lyapunov functions is a standard (if not the only) method to tackle these problems for large-scale systems.

Within the framework of coupled systems on digraphs, construction of a global Lyapunov function V for the coupled system (a difficult task) can be aided by Lyapunov functions Vi for isolated vertex systems, which are typically small scale and well studied. I will present a recent approach that uses results from graph theory to systematically build a Lyapunov function V for the large-scale system from Vi of the vertex systems.

I will introduce some elementary results from graph theory such as digraphs and their weight matrices, equivalence of irreducibility of a weight matrix and strong connectedness of the digraph, the Laplacian matrix, spanning trees, and Kirchhoff Matrix-Tree Theorem, as well as unicyclic graphs and the Tree-Cycle Identity. Then I will describe a recent result of Guo-Li-Shuai [2-5] on the construction of global Lyapunov functions for coupled systems on digraphs.

I will show how the graph-theoretic method can be applied analyze two large-scale models: a multi-group SIR model and a multi-city model for Dengue fever.

Bibliography

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